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Large Diffusivity and Asymptotic Behavior in Parabolic Systems*

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For systems of reaction–diffusion equations with Neumann boundary conditions, it is shown that the solutions are asymptotic to the solutions of an ordinary differential equation if the diffusivity is large. The methods apply also to reaction–diffusion systems with time delays. © 1986 Academic Press, Inc.

1. INTRODUCTION AND STATEMENT OF RESULTS

Many models of chemical, biological, and ecological problems involve systems of reaction–diffusion equations of the form

$$\begin{aligned} \partial u/\partial t &= D\,\Delta u + f(u) && \text{in } \Omega \\ \partial u/\partial n &= 0 && \text{in } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^n$ with $\partial\Omega$ smooth, $u \in \mathbb{R}^n$, $D = \operatorname{diag}(d_1, ..., d_N)$, where each $d_j > 0$ is a constant and $f\colon \mathbb{R}^N \to \mathbb{R}^N$ is a C^2-function. Other types of boundary conditions may also occur.

In recent years there have been many investigations devoted to the study of stable patterns for Eq. (1.1), that is, stable solutions which are spatially dependent (see, for example, [1 – 8]). For the understanding of how stable patterns are created, it is obviously of interest to characterize those situations for which stable patterns do not exist and, even more particularly, those systems for which the flow is essentially determined by the ordinary differential equation (ODE)

$$du/dt = f(u). \tag{1.2}$$

This latter problem has been investigated by Conway, Hoff, and Smoller

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[9] under the assumption that there is an *invariant region* Σ for Eq. (1.1). A set $\Sigma \subset \mathbb{R}^N$ is an invariant region for Eq. (1.1) if, for any initial data u_0 with $u_0(x) \in \Sigma$ for all $x \in \Omega$, one has the solution $u(t, x)$ through u_0 at $t = 0$ also in Σ for all $x \in \Omega$. In [10, 2], it is shown that an invariant region Σ for Eq. (1.1) must be a rectangle in $\mathbb{R}^N$ if all the diffusion coefficients d_j are distinct. If the diffusion coefficients are equal, then Σ can be any convex set which is positively invariant for Eq. (1.2).

Suppose Σ is an invariant region for Eq. (1.1), $M = \sup\{|\partial f(u)/\partial u|: u \in \Sigma\}$, $-\lambda$ is the first nonzero eigenvalue of the Laplacian with homogeneous Neumann boundary conditions on Ω, $d = \min(d_1, ..., d_N)$, and $\sigma = d\lambda - M$. In [9], it was shown that $\sigma > 0$ implies the solutions of Eq. (1.1) with initial data in Σ approach a solution of the ODE (1.2) as $t \to \infty$. Since λ is inversely proportional to the squared diameter of Ω, the hypothesis $\sigma > 0$ says that diffusion on the domain is fast relative to the reaction term f. In fact, the estimates in [9] show that the spatial inhomogeneities are quickly damped out if σ is very large.

As remarked earlier, the hypothesis that Σ be an invariant region for Eq. (1.1) severely limits the types of equations that can be considered. In fact, as pointed out by Smoller [2, p. 212], the property that Σ is an invariant region for Eq. (1.1) is not continuous with respect to the d_j. In fact, if Σ is a convex positively invariant set for the ODE (1.2) and each $d_j = 1$, $j = 1, 2, ..., N$, then Σ is invariant for Eq. (1.1). However, if Σ is not a rectangle then it is not invariant for Eq. (1.1) unless the d_j remain equal.

It is the purpose of this paper to begin an investigation of the behavior of the solutions of Eq. (1.1) when the constant $d\lambda$ is large and Eq. (1.1) may not have an invariant region in $\mathbb{R}^N$. The methods will use properties of the flow defined by Eq. (1.1) in function space and will be applicable to situations where the equation with no diffusion is a retarded functional differential equation or a differential equation with delays. In this latter case it is almost impossible to have an equation with an invariant region.

We now describe the results in some detail. Let $X = L^2(\Omega, \mathbb{R}^N)$, $D(A) = \{\phi \in W^{2,2}(\Omega, \mathbb{R}^N): \partial\phi/\partial n = 0 \text{ on } \partial\Omega\}$, $A = -\Delta: D(A) \to X$. In the usual way, one defines the fractional power spaces X^α using the operator A. If $n \leqslant 3$, $\frac{3}{4} < \alpha < 1$, then it is known that $X^\alpha \subset W^{1,2}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ with continuous inclusion. One can then show (see, for example, Henry [10, p. 75]) that, for any $u_0 \in X^\alpha$, $\alpha > \frac{3}{4}$, there is a unique solution $u(t, \cdot, u_0) \in X^\alpha$ of Eq. (1.1) through u_0 at $t = 0$ which is continuous in t, u_0.

A set $\mathscr{A} \subset X^\alpha$ is a *compact attractor for Eq.* (1.1) if $\mathscr{A}$ is compact, invariant, and there is a neighborhood U of $\mathscr{A}$ such that the ω-limit set of U is $\mathscr{A}$. By the ω-limit set $\omega(U)$ of U, we mean

$$\omega(U) = \bigcap_{\tau \geqslant 0} Cl \bigcup_{t \geqslant \tau} u(t, \cdot, U).$$

In a similar way one can define a *compact attractor in* $\mathbb{R}^N$ for the ODE (1.2).

If $\mathscr{A}$ is a compact attractor for the ODE (1.2) then it can be considered as a subset of the constant functions in X^α and it will be a compact invariant set for the PDE (1.1). However, it need not be an attractor without some conditions on D and Ω. If $d = \min\{d_j, j = 1, 2, ..., N\}$ and $-\lambda$ is the first nonzero eigenvalue of Δ with Neumann conditions, then the main result of the paper is the following:

THEOREM 1.1. *Suppose $\mathscr{A}$ is a compact attractor for the ODE (1.2). Then there exists a $\delta > 0$ such that $\mathscr{A}$ considered as a subset of the constant functions in X^α, $\alpha > \frac{3}{4}$, $n \leqslant 3$, is a compact attractor for the PDE (1.1) if $d\lambda > \delta$. More precisely, there is a neighborhood V of $\mathscr{A}$ in X^α and constants $K > 0$, $c > 0$ such that, for any $u_0 \in V$, the solution $u(t, \cdot, u_0)$ of Eq. (1.1) through u_0 at $t = 0$ satisfies*

$$|u(t, \cdot, u_0) - \bar{u}(t)|_{X^\alpha} \leqslant Ke^{-ct}, \qquad t \geqslant 0,$$

where $\bar{u}(t) = |\Omega|^{-1} \int_\Omega u(t, x, u_0)\, dx$ and $\bar{u}(t)$ satisfies the equation

$$d\bar{u}(t)/dt = f(\bar{u}(t)) + g(t, u_0),$$

where $|g(t, u_0)| \leqslant Ke^{-ct}$, $t \geqslant 0$.

The technical part of the theorem states that the solution $u(t, \cdot, u_0)$ approaches its average value $\bar{u}(t)$ exponentially as $t \to \infty$ in the space X^α. Since $\alpha > \frac{3}{4}$, this implies, in particular, that the solution approaches $\bar{u}(t)$ exponentially as $t \to \infty$ in $L^\infty(\Omega, \mathbb{R}^N)$. The conclusions in Theorem 1.1 are the same as the ones in Conway, Hoff, and Smoller [9] mentioned above.

The proof of the theorem is given in Section 2 and uses an elementary property of Liapunov functions for ODEs and a special decomposition of Eq. (1.1). In Section 3 we consider invariant regions and show how the method presented here gives the same qualitative results as in [9], but the rates of decay are not as sharp. Generalizations to functional differential equations are given in Section 4.

2. PROOF OF THEOREM 1.1

If $M \subset \mathbb{R}^N$ is a given set and $x \in \mathbb{R}^N$ is given, we let $d(x, M)$ denote the distance from x to M. Suppose $\mathscr{A}$ is a compact attractor for the ODE (1.2). From Yoshizawa [11, p. 111], there are a neighborhood U of $\mathscr{A}$ and a Lipschitz continuous function $V: U \to \mathbb{R}$ such that, for any $x \in U$,

(i) $V(x)=0$ if $x \in \mathscr{A}$;

(ii) $a(d(x, \mathscr{A})) \leqslant V(x) \leqslant b(d(x, \mathscr{A}))$ where $a(r)$ is continuous, nondecreasing, $a(r)>0$ if $r>0$, and $b(r)$ is continuous, $b(0)=0$.

(iii) $\dot{V}_{(1.2)}(x) \leqslant -V(x)$ where $\dot{V}_{(1.2)}(x)=\overline{\lim}_{h\to 0}\, h^{-1}[V(u(h, x))-V(x)]$ with $u(t, x)$ being the solution of Eq. (1.2) through x at $t=0$.

In the following, for any $c>0$ we let $V_c=\{x \in U\colon V(x)<c\}$, $\bar{V}_c=Cl V_c$. From property (ii) above, $\bar{V}_c$ is compact for any $c>0$.

Let $W \subset X^\alpha$ be the linear subspace consisting of the constant functions, $X^\alpha = W \oplus W_\alpha^\perp$, $u=v+w$, where $v \in W$, $w \in W_\alpha^\perp$,

$$v=|\Omega|^{-1} \int_\Omega u(x)\, dx, \qquad \int_\Omega w(x)\, dx=0. \tag{2.1}$$

We can identify W with $\mathbb{R}^N$ and therefore will consider v as an element of W as well as a vector in $\mathbb{R}^N$.

Suppose $u(t, \cdot)$ is a solution of Eq. (1.1) and let $u(t, \cdot)=v(t)+w(t, \cdot)$, $v(t) \in W$, $w(t, \cdot) \in W_\alpha^\perp$. Then

$$\begin{aligned} dv/dt &= P(v, w) \\ \partial w/\partial t &= D\, \Delta w + Q(v, w), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} P(v, \phi) &= |\Omega|^{-1} \int_\Omega f(v+\phi(x))\, dx \\ Q(v, \phi)(x) &= f(v+\phi(x)) - |\Omega|^{-1} \int_\Omega f(v+\phi(y))\, dy. \end{aligned} \tag{2.3}$$

Observe that

$$P(v, 0)=f(v), \qquad Q(v, 0)=0. \tag{2.4}$$

Also, for $\psi \in W_\alpha^\perp$, we have

$$\begin{aligned} [Q_\phi(v, \phi)\, \psi](x) &= f'(v+\phi(x))\, \psi(x) - |\Omega|^{-1} \int_\Omega f'(v+\phi(y))\, \psi(y)\, dy \\ &= [f'(v+\phi(x)) - f'(v)]\, \psi(x) + f'(v)\, \psi(x) \\ &\quad - |\Omega|^{-1} \int_\Omega [f'(v+\phi(y)) - f'(v)]\, \psi(y)\, dy, \end{aligned}$$

since $\int_\Omega \psi = 0$. Since $\alpha > \frac{3}{4}$, $\phi \in X^\alpha$ implies $\phi \in L^\infty(\Omega, \mathbb{R}^N)$ and there is a constant k such that $|\phi|_{L^\infty} \leqslant k\, |\phi|_{X^\alpha}$. Let $M_c = \sup\{|f'(v)|\colon v \in \bar{V}_c\}$. For any $c_1 < c$, there is a $\delta > 0$ such that $|\phi|_{X^\alpha} < \delta$, $v \in V_{c_1}$ implies $v+\phi(x) \in \bar{V}_c$ for

$x \in \Omega$. Since f is a C^2-function, it follows that there is a constant N_c such that

$$|f'(v+\phi(x)) - f'(v)| \leqslant N_c \, |\phi|_{L^\infty} \leqslant N_c k \, |\phi|_{X^\alpha}$$

$v \in V_{c_1}$, $|\phi|_{X^\alpha} < \delta$. Therefore,

$$|[Q_\phi(v, \phi)\, \psi](x) - f'(v)\, \psi(x)| \leqslant 2k^2 N_c \, |\phi|_{X^\alpha} \, |\psi|_{X^\alpha}$$

which implies

$$|Q_\phi(v, \phi)\, \psi - f'(v)\, \psi|_X \leqslant 2k^2 N_c \, |\Omega|^{1/2} \, |\phi|_{X^\alpha} \, |\psi|_{X^\alpha}. \tag{2.5}$$

Relation (2.5) and the triangle inequality imply that

$$\begin{gathered} |Q(v, \phi)| \leqslant \theta \, |\phi|_{X^\alpha} \\ \theta = M_c k + 2k^2 N_c \, |\Omega|^{1/2} \, \delta \end{gathered} \tag{2.6}$$

in the set $v \in V_{c_1}$, $|\phi|_{X^\alpha} < \delta$.

Let $T(t)$ be the semigroup on $W_\alpha^\perp$, $0 \leqslant \alpha < 1$, generated by the equation

$$\begin{aligned} \partial w/\partial t &= D\, \Delta w \qquad && \text{in } \Omega \\ \partial w/\partial n &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Now, fix $\alpha > \frac{3}{4}$. There is a constant $k_1 > 0$ such that

$$\begin{aligned} |T(t)\, w|_{X^\alpha} &\leqslant k_1 e^{-d\lambda t} \, |w|_{X^\alpha}, \qquad && t \geqslant 0,\ w \in W_\alpha^- \\ |T(t)\, w|_{X^\alpha} &\leqslant k_1 e^{-d\lambda t} t^{-\alpha} \, |w|_X, && t \geqslant 0,\ w \in W^\perp. \end{aligned} \tag{2.7}$$

We can now use the variation of constants formula to rewrite the initial value problem for Eq. (2.2) as

$$\begin{aligned} dv/dt &= f(v) + [P(v, w) - f(v)], \qquad v(0) = v_0, \\ w(t) &= T(t)\, w_0 + \int_0^t T(t-s)[Q(v(s), w(s))]\, ds. \end{aligned} \tag{2.8}$$

For any $c_1 < c$, choose δ as before so that $|\phi|_{X^\alpha} < \delta$, $v \in V_{c_1}$ implies $v + \phi(x) \in \bar{V}_c$. Choose a constant σ, $0 < \sigma < \lambda$, let k_2 be the Lipschitz constant for V on $\bar{V}_c$, $L = \int_0^\infty s^{-\alpha} e^{-(1-\sigma/\lambda)s}\, ds$ and let $V(v) \geqslant p > 0$ for $v \in \bar{V}_c \setminus V_{c_1}$. With the constants as in Eq. (2.6), choose $d\lambda > 0$ and $\eta < \delta$ so that

$$\begin{gathered} \zeta \stackrel{\text{def}}{=} k_1 L \theta (d\lambda)^{\alpha - 1} < 1 \\ p - k_2 M_c k \eta > 0. \end{gathered} \tag{2.9}$$

Let $v(t)$, $w(t)$ satisfy Eq. (2.8). If $v(s) \in V_{c_1}$, $|w(s)|_{X^\alpha} < \eta$ for $0 \leqslant s \leqslant t$, then relation (2.6) implies that

$$\begin{aligned}
\dot{V}(v(t)) &\leqslant -V(v(t)) + k_2\,|P(v(t), w(t)) - f(v(t))| \\
&\leqslant -V(v(t)) + k_2 M_c k\,|w(t)|_{X^\alpha} \\
&\leqslant -V(v(t)) + k_2 M_c k\eta
\end{aligned}$$

$$z(t) \leqslant k_1 e^{-d(\lambda-\sigma)t} z(0) + k_1\theta \int_0^t (t-s)^{-\alpha} e^{-d(\lambda-\sigma)(t-s)} z(s)\,ds,$$

where $z(t) = |w(t)|_{X^\alpha} \exp d\sigma t$.

If $y(t) = \sup\{z(s), 0 \leqslant s \leqslant t\}$, then

$$z(t) \leqslant k_1 e^{-d(\lambda-\sigma)t} z(0) + \zeta y(t).$$

This implies that $y(t) \leqslant k_1(1-\zeta)^{-1} z(0)$. This implies that

$$|w(t)|_{X^\alpha} \leqslant \frac{k_1}{1-\zeta}\, e^{-d\sigma t}\,|w_0|_{X^\alpha}.$$

Thus, we see that

$$\begin{aligned}
&\dot{V}(v(t)) \leqslant -V(v(t)) + k_2 M_c k\eta \\
&|w(t)|_{X^\alpha} \leqslant \frac{k_1}{1-\zeta}\, e^{-d\sigma t}\,|w_0|_{X^\alpha}
\end{aligned} \tag{2.10}$$

if it is assumed that $v(s) \in V_{c_1}$, $|w(s)|_{X^\alpha} < \eta$.

Now choose δ_1 so that $k_1(1-\zeta)^{-1}\delta_1 < \eta$. If $v_0 \in V_{c_1}$, $|w_0|_{X^\alpha} < \delta_1$ then relations (2.9), (2.10) imply that $v(t) \in V_{c_1}$, $|w(t)|_{X^\alpha} < \eta$ for all $t \geqslant 0$.

Relation (2.10) also implies that $|w(t)|_{X^\alpha}$ approaches zero exponentially as $t \to \infty$. Thus, the ω-limit set of every solution of Eq. (2.2) with initial value $v_0 \in V_{c_1}$ and $|w_0|_{X^\alpha} < \delta_1$ must have $w = 0$; that is, the ω-limit set of any solution of Eq. (1.1) with $u_0 = v_0 + w_0$ satisfying the above condition must lie in the set $W \cap V_{c_1}$. Furthermore

$$dv(t)/dt = f(v(t)) + [P(v(t), w(t)) - f(v(t))]$$

and the second term approaches zero exponentially as $t \to \infty$. Therefore, the limit set of v must be a union of invariant sets of $dv/dt = f(v)$ which belong to V_{c_1} (see Yoshizawa [11]). However, all such invariant sets must belong to $\mathscr{A}$. This completes the proof of the theorem.

3. The Case of an Invariant Region

If Eq. (1.1) has an invariant region Σ in $\mathbb{R}^N$, and $v \in \Sigma$, $\phi \in W_\alpha^\perp$, $v + \phi(x) \in \Sigma$ for $x \in \Omega$, then Conway, Hoff, and Smoller [9] prove a more global version of Theorem 1.1. In fact, assuming the initial data $u_0 \in C^{k,1}(\Omega, \mathbb{R}^N)$, $u_0(x) \in \Sigma$ for $x \in \Omega$, they prove that the solution $u(t, x)$ through u_0 satisfies $|\nabla_x u(t, \cdot)|_{L^2}$, $|u(t, \cdot) - \bar{u}(t)|_{L^\infty} \to 0$ as $t \to \infty$ exponentially with exponent σ. To obtain the L^2 estimate on $\nabla_x u$ and an L^2 estimate on $u(t, \cdot) - \bar{u}(t)$ is not difficult using simple integration by parts and the variational characterization of the first eigenvalue of Δ. The L^∞ estimate uses more sophisticated properties of parabolic equations and the reader is referred to [9] for details.

The method used in the previous section can be applied to this case but the exponential estimate is not as precise. We briefly indicate how this can be accomplished. If $u_0 \in X^\alpha$, $\alpha > \frac{3}{4}$, and $u_0(x) \in \Sigma$ for $x \in \Omega$, then the solution $u(t, x)$ through u_0 remains in Σ for all $t \geqslant 0$. If $\phi \in W_\alpha^\perp$, $v + \phi(x) \in \Sigma$ for $x \in \Omega$, then one easily shows that $|Q_\phi(v, \phi)| \leqslant 2\, |\Omega|^{1/2}\, kM$, where k is such that $|\phi|_{L^\infty} \leqslant k\, |\phi|_{X^\alpha}$ and $M = \sup\{|f'(v)|,\ v \in \Sigma\}$. Using the variation of constants formula

$$w(t) = e^{D\Delta t} w_0 + \int_0^t e^{D\Delta(t-s)} Q(v(s), w(s))\, ds$$

and letting $k_3 = k_1\, |\Omega|^{1/2}$, one obtains

$$\begin{aligned} |w(t)|_{X^\alpha} &\leqslant k_1 e^{-d\lambda t}\, |w_0|_{X^\alpha} \\ &\quad + 2k_3 kM \int_0^t e^{-d\lambda(t-s)} (t-s)^{-\alpha}\, |w(s)|_{X^\alpha}\, ds. \end{aligned}$$

Let $z(t) = |w(t)|_{X^\alpha} \exp \beta t$, $d\lambda - \beta > 0$, $y(t) = \sup_{0 \leqslant s \leqslant t} z(s)$. Then

$$\begin{aligned} z(t) &\leqslant k_1 z(0) + 2k_3 kM \int_0^t e^{-(d\lambda - \beta)(t-s)} (t-s)^{-\alpha}\, z(s)\, ds \\ &\leqslant k_1 z(0) + 2k_3 kM \int_0^\infty e^{-(d\lambda - \beta)s} s^{-\alpha}\, ds\, y(t). \end{aligned}$$

It is clear that one can choose β, d so that the coefficient θ of $y(t)$ is < 1. Then,

$$|w(t)|_{X^\alpha} \leqslant k_1 (1 - \theta)^{-1}\, e^{-\beta t}\, |w_0|_X, \qquad t \geqslant 0.$$

This estimate implies that $|w(t)|_{X^\alpha}$ and so $|w(t)|_{L^\infty} \to 0$ exponentially as $t \to \infty$. This also implies that $P(v(t), w(t)) - f(v(t)) \to 0$ exponentially and we obtain all of the conclusions of Theorem 1.1. Of course, the exponential estimate is not a good one.

4. Functional Differential Equations

Suppose $r > 0$ is a given constant, $f: C([-r, 0], \mathbb{R}^N) \to \mathbb{R}^N$ is a given C^2-function, and consider the equations

$$\begin{aligned} \partial u(t, x)/\partial t &= D\,\Delta u(t, x) + f(u_t(\cdot, x)) && \text{in } \Omega \\ \partial u(t, x)/\partial n &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

where $u_t(\theta, x) = u(t + \theta, x)$, $\theta \in [-r, 0]$, $x \in \Omega$.

As before, let $X = L^2(\Omega, \mathbb{R}^N)$ define the spaces X^α and choose $\frac{3}{4} < \alpha < 1$. For any $\phi \in C([-r, 0], X^\alpha)$ with $\partial\phi(\theta, x)/\partial n = 0$, $-r \leqslant \theta \leqslant 0$, one can define a solution $u(t, x, \phi)$ of Eq. (4.1) on an interval $-r \leqslant t < \alpha$, $\alpha > 0$, with $u(\theta, x, \phi) = \phi(\theta, x)$, $-r \leqslant \theta \leqslant 0$. This solution $u(t, \cdot, \phi)$ will be in X^α, it will be continuous in t, ϕ, and continuously differentiable in ϕ. The proof of this fact follows along the lines of Henry [10] or Travis and Webb [11].

If we let $T(t, \phi) \in C([-r, 0], X^\alpha)$ be defined by $T(t, \phi)(\theta) = u(t + \theta, \cdot, \phi)$, $-r \leqslant \theta \leqslant 0$, then $T(t, \cdot)$ defines a local semiflow on $C([-r, 0], X^\alpha)$. Positive, negative, and complete orbits are defined in the usual way as well as α-limit sets, ω-limit sets, invariant sets, and compact attractors.

If $\mathscr{A} \subset C([-r, 0], \mathbb{R}^N)$ is a compact attractor for the retarded functional differential equation (RFDE),

$$dv(t)/dt = f(v_t(\cdot)) \tag{4.2}$$

then there is a Liapunov function V defined in a neighborhood of $\mathscr{A}$ and satisfying conditions (i)–(iii) of Section 2 (see Yoshizawa [11]).

If $\psi \in C([-r, 0], \mathbb{R}^N)$ and $v(t, \psi)$ is the solution of Eq. (4.2) through ψ at $t = 0$ and $\tilde{\psi}(\theta)(x) = \psi(x)$, $-r \leqslant \theta \leqslant 0$, $x \in \Omega$, then $\tilde{\psi} \in C([-r, 0], X^\alpha)$ and $u(t, x, \tilde{\psi}) = v(t, \psi)$, $x \in \Omega$, is the solution of Eq. (4.1) through $\tilde{\psi}$. Thus, every solution of the RFDE (4.2) is a solution of Eq. (4.1).

To obtain the analog of Theorem 1.1 for Eq. (4.1), we make the decomposition of X^α as $X^\alpha = U \oplus U_\alpha^\perp$ as was done in Section 2, where U is identified with $\mathbb{R}^N$ and $u = v + w$, $v \in U$, $w \in U_\alpha^\perp$ implies $v = |\Omega|^{-1} \int_\Omega u(x)\,dx$. This decomposition of X^α induces in a natural way a decomposition of $C([-r, 0], X^\alpha)$ as

$$C([-r, 0], X^\alpha) = C([-r, 0], \mathbb{R}^N) \oplus C([-r, 0], U_\alpha^\perp)$$

$$\phi = \xi + \psi, \qquad \xi(\theta) = |\Omega|^{-1} \int_\Omega \phi(\theta, x)\,dx.$$

If u is a solution of Eq. (4.1) with initial value u_0 and

$$u(t, x) = v(t) + w(t, x)$$

$$v(t) = |\Omega|^{-1} \int_{\Omega} u(t, x)\, dx$$

then it follows from Eq. (4.1) that

$$\dot{v}(t) = P(v_t, w_t) \tag{4.3}$$

$$\dot{w}(t) = D\, \Delta w(t) + Q(v_t, w_t), \tag{4.4}$$

where the initial values v_0, w_0 of v, w are given by

$$\begin{aligned} v_0(\theta) &= |\Omega|^{-1} \int_{\Omega} u_0(\theta, x)\, dx \\ w_0(\theta, x) &= u_0(\theta, x) - v_0(\theta), \qquad -r \leqslant \theta \leqslant 0, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} P(\xi, \psi) &= |\Omega|^{-1} \int_{\Omega} f(\xi + \psi(\cdot, x))\, dx \\ Q(\xi, \psi)(x) &= f(\xi + \psi(\cdot, x)) - P(\xi, \psi) \\ \xi \in C([-r, 0], \mathbb{R}^N), &\qquad \psi \in C([-r, 0], U_\alpha^\perp). \end{aligned} \tag{4.6}$$

Equation (4.4) with initial data w_0 can be written as

$$\begin{aligned} w(t, x) &= e^{DAt} w_0(0, x) + \int_0^t e^{DA(t-s)} Q(v_s, w_s)\, ds, &\qquad t \geqslant 0 \\ w(t, x) &= w_0(t, x), &\qquad t \leqslant 0, \end{aligned}$$

or, for $-r \leqslant \theta \leqslant 0$,

$$\begin{aligned} w(t+\theta, x) &= e^{DA(t+\theta)} w_0(0, x) + \int_0^{t+\theta} e^{DA(t+\theta-s)} Q(v_s, w_s)\, ds, &\qquad t+\theta \geqslant 0, \\ w(t+\theta, x) &= w_0(t+\theta, x), &\qquad t+\theta < 0 \end{aligned}$$

If we define the $n \times n$ matrix function $X(\theta)$ and semigroup $S(t)$ by $X_0(\theta) = 0$ for $\theta < 0$, $X_0(0) = I$, the identity

$$\begin{aligned} [S(t)\, w_0](0, x) &= e^{DA(t+\theta)} w_0(0, x), &\qquad t+\theta \geqslant 0 \\ &= w_0(t+\theta, x), &\qquad t+\theta < 0 \end{aligned}$$

then the above formula for the solution w becomes

$$w_t(\theta, x) = [S(t)\, w_0](\theta, x) + \int_0^t [e^{DA(t-s)}X_0](\theta)\, Q(v_s, w_s)\, ds \tag{4.7}$$

for all $t \geq 0$, $-r \leq \theta \leq 0$. We will write this last equation as

$$w_t = S(t)\, w_0 + \int_0^t e^{DA(t-s)} X_0 Q(v_s, w_s)\, ds, \qquad t \geq 0 \tag{4.8}$$

always remembering that it is evaluated as in Eq. (4.7).

Using the estimates (2.7), Eqs. (4.3), (4.8), and the same type of arguments as in the proof of Theorem 1.1, one obtains the following result:

THEOREM 4.1. *Suppose $\mathscr{A} \subset C([-r, 0], \mathbb{R}^N)$ is a compact attractor for the RFDE* (4.2). *Then there is a $\delta > 0$ such that $\mathscr{A}$ considered as a subset of $C([-r, 0], X^\alpha)$, $\alpha > \frac{3}{4}$, is a compact attractor for Eq.* (4.1) *if $d\lambda > \delta$. More precisely, there is a neighborhood V of $\mathscr{A}$ in $C([-r, 0], X^\alpha)$ and constants $K > 0$, $c > 0$ such that, for any $u_0 \in V$, the solution $u(t, x, u_0)$, $t \geq -r$, of Eq.* (4.1) *with $u(\theta, x, u_0) = u_0(\theta, x)$, $-r \leq \theta \leq 0$ satisfies*

$$|u(t, \cdot, u_0) - \bar{u}(t)|_{X^\alpha} \leq Ke^{-ct}, \qquad t \geq 0,$$

where $\bar{u}(t) = |\Omega|^{-1} \int_\Omega u(t, x)\, dx$, $t \geq -r$, and $\bar{u}(t)$ satisfies the equation

$$d\bar{u}(t)/dt = f(\bar{u}_t) + g(t, u_0), \qquad t \geq 0,$$

where

$$|g(t, u_0)| \leq Ke^{-ct}, \qquad t \geq 0.$$

5. EXAMPLES

Examples in ordinary differential equations are very easy to obtain. Conway, Hoff, and Smoller [9] have several interesting ones for the case in which there is an invariant region Σ. Any two species Volterra–Lotka model for which the ODE's have a unique stable limit cycle in the positive quadrant $\mathbb{R}^2_+$ would have a compact attractor $\mathscr{A}$ in $\mathbb{R}^2_+$. Therefore, this model with large diffusion would have the same attractor $\mathscr{A}$ in the function space.

As remarked earlier, it is almost impossible to have an invariant region when the equation without diffusion is an RFDE. Therefore, we give an example illustrating an implication of the theory for this case.

Consider the scalar differential difference equation

$$\dot{v}(t) = -(\pi/2 + \mu)\, v(t-1)[1 + v(t)] \tag{5.1}$$

in a neighborhood of $\mu = 0$, $v = 0$. Zero is always a solution of this equation. Also, for $\mu < 0$, the origin is asymptotically stable. At $\mu = 0$, the linearized equation has two eigenvalues on the imaginary axis with the remaining ones having negative real parts. For $\mu > 0$ there is a Hopf bifurcation to a stable periodic orbit. (See, for example, Hale [13], Chow and Mallet-Paret [14], Stech [15].) Therefore, there is a $\mu_0 > 0$ and a neighborhood W of zero in $C([-1, 0], \mathbb{R})$ such that Eq. (5.1) has a compact attractor of $\mathscr{A}_\mu$ in W and $\mathscr{A}_\mu = \{0\}$ for $-\mu_0 \leqslant \mu \leqslant 0$ and $\mathscr{A}_\mu = W^u_\mu(0) \cup \gamma_\mu$ for $0 < \mu \leqslant \mu_0$, where γ_μ is the periodic orbit obtained from the Hopf biburcation and $W^u_\mu(0)$ is the unstable manifold of the zero solution.

Theorem 4.1 implies that $\mathscr{A}_\mu$ is an attractor for the equation

$$\begin{aligned} \partial u(t, x)/\partial t &= d\,\Delta u(t, x) - (\pi/2 + \mu)\, u(t-1, x)[1 + u(t, x)] && \text{in } \Omega \\ \partial u(t, x)/\partial n &= 0 && \text{on } \partial\Omega \end{aligned}$$

in $C([-1, 0], X^\alpha)$, $\alpha > \frac{3}{4}$, if $d\lambda > \delta$, where δ is sufficiently large. Furthermore, the estimates in Theorem 4.1 show, in particular, that the orbit γ_μ is asymptotically orbitally stable in $C([-1, 0], X^\alpha)$ if $d\lambda > \delta$.

It is clear that the above remarks remain valid for any RFDE in a neighborhood of a Hopf bifurcation. In particular, a stable Hopf bifurcation remains stable if $d\lambda$ is sufficiently large which is a result previously obtained by Yoshida [16].

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